Nonlinear-relaxation-time and quasideterministic-theory approaches to characterize the decay of unstable states

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We present the relationship between nonlinear-relaxation-time (NLRT) and quasideterministic approaches to characterize the decay of an unstable state. The universal character of the NLRT is established. The theoretical results are applied to study the dynamical relaxation of the Landau model in one and n variables and also a laser model.

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I. INTRODUCTION

The study of the transient behavior of nonequilibrium systems has become an interesting topic in stochastic dynamics [1–9], particularly the decay of the unstable states, in which the fluctuations play a very important role. Two approaches have essentially been proposed for the characterization of such unstable states. One is the so-called first-passage-times (FPT) distribution [3–6], and the other one is the nonlinear relaxation times (NLRT) [7–9]. The FPT defines a random variable which is the time t necessary to cross a given boundary. It is mainly based on the quasideterministic (QD) approach [1–5]. The NLRT is directly associated with general relaxation processes of certain quantities, such as the moments of the relevant stochastic variable, from arbitrary initial conditions to the corresponding steady states. The QD approach is a good approximation because it gives a precise physical picture of the mechanism responsible for the decay of the unstable state. In fact this approach considers the fluctuations around the unstable initial state as the driving mechanism to initiate the relaxation. It is only at this state that the fluctuations are important, in such a way that the relaxation of the system, after its stochastic beginning, is mainly deterministic until it reaches the steady state. Its essential physical meaning is that the fluctuations change the initial state of the system in the neighborhood of the unstable state and then the deterministic motion drives the system out of this state.

On the other hand, some results [10–12] characterizing the decay of the unstable state obtained by the NLRT approach are similar to those obtained by using FPT distributions through the QD approach. However, the relationship between the NLRT and QD approach, although intuitively used, has not been discussed in a formal way. It is the purpose of this paper to show how they are connected. We shall use a method which reduces the definition of the NLRT to a quadrature. This method will allow us to study, in a natural way, the different types of unstable models, whether linear or nonlinear, which have also been analyzed by other mechanisms [7,8,10–12].

In Sec. II we introduce the general definition of an unstable state and we analyze the mechanism of the QD approach. In Sec. III the connection between the NLRT and QD approach is analyzed and expressions of universal character of the NLRT for the decay of an unstable state is established. As an application of the theoretical results, the Landau and laser models are studied in Sec. IV. Conclusions are finally given in Sec. V.

II. GENERAL DEFINITION OF AN UNSTABLE STATE AND QD APPROACH

A. Definition of an unstable state

The characterization of an unstable state is made in general from the deterministic equation of the square $r=x^2$ of the relevant variable x,

$$\dot{r}=v(r) , \tag{2.1}$$

where the function $v(r)$ is such that it has two roots. One root is at $r=0$, which is the unstable state, then $v'(r)|_{r=0}>0$, and the other root is at $r=r_{\rm st}$, which corresponds to the stable state, such that $v'(r)|_{r=r_{\rm st}}<0$. Therefore the most general expression for the function $v(r)$ will be

$$v(r)=\frac{r}{f(r)}(r_{\rm st}-r) , \tag{2.2}$$

with $f(r)$ a polynomial function different from zero. It can be written in a general form $f(r)=C_0+rg(r)$, where $C_0=r_{\rm st}/2a$ is a constant and $g(r)>0$ is a polynomial.

The QD approach assumes that the initial condition $r_0=r(0)$ of (2.1) is a stochastic variable. It takes into account initial fluctuations of the system.

B. The QDT approach

We will now review the main characteristics of the QD approach. This analysis will give us the results which are necessary to make the connection with the NLRT approach.

First of all let us recall that in the study of transient situations it is usual to assume a description in terms of a Langevin-type equation for the relevant variable. In the case of one variable it reads

$$\dot{x}(t)=v(x)+g(x)\xi(t) , \tag{2.3}$$

where $\xi(t)$ is the stochastic force or noise.

The basic ingredients of the QD approach appear in the linear approximation of Eq. (2.3) with additive noise, namely

$$\dot{x}=ax+\xi(t) . \tag{2.4}$$

$\xi(t)$ is a Gaussian white noise with zero mean and correlation function

$$\langle \xi(t)\xi(t')\rangle=2D\delta(t-t') , \tag{2.5}$$

where D is the intensity of the noise.

The formal solution of (2.4) can be written such that

$$x(t)=h(t)e^{at} , \tag{2.6}$$

where

$$h(t)=\int_0^t e^{-at'}\xi(t')dt' . \tag{2.7}$$

In Eq. (2.6) $h(t)$ will play the role of a stochastic initial condition, because for long times, $at \gg 1$, it becomes a Gaussian random variable, $h(\infty)=h$, with the first two moments given by $\langle h\rangle=0$ and $\langle h^2\rangle=D/a=\sigma$. Therefore, the probability density of the variable h is

$$P(h)=\frac{2\alpha}{\sqrt{\pi}}e^{-\alpha^2h^2} , \tag{2.8}$$

where $\alpha^2=1/2\sigma$.

In the case of n variables $(x_1,x_2,\ldots,x_n)$ the Langevin equation for each variable x_i is approximated by

$$\dot{x}_i(t)=ax_i+\xi_i(t) , \tag{2.9}$$

where $\xi_i(t)$ has zero mean value and correlation function

$$\langle \xi_i(t)\xi_j(t')\rangle=2D\delta_{ij}\delta(t-t'), \quad i,j=1,2,\ldots,n . \tag{2.10}$$

Now the formal solution of (2.9) for each x_i is the same as (2.6), but

$$h_i(t)=\int_0^t e^{-at'}\xi_i(t')dt' . \tag{2.11}$$

Again $h_i(t)$ plays the role of a stochastic Gaussian initial condition with variance equal to $\langle h_i^2(\infty)\rangle=\langle h_i^2\rangle=D/a=\sigma$. Now the probability for the modulus h reads

$$P(h)=\frac{2\alpha^n}{\Gamma(n/2)}h^{n-1}e^{-\alpha^2h^2} . \tag{2.12}$$

III. UNIVERSAL CHARACTER OF THE NLRT OF AN UNSTABLE STATE

In this section we will give two general expressions for the NLRT which will be obtained by a QD approach. These expressions will allow us to characterize the transient dynamics of an unstable state, through the evolution of a statistical moment. Here we define the NLRT associated with the average of the variable r. For this quantity the NLRT is [8,9]

$$T=\int_0^\infty \frac{\langle r(t)\rangle-\langle r\rangle_{\rm st}}{\langle r\rangle_0-\langle r\rangle_{\rm st}}dt , \tag{3.1}$$

where we will assume that $r(0)=0$, which means that the initial condition is fixed at the unstable state.

The QD approach for the definition (3.1) is obtained from the deterministic equation (2.1). The essential point is that the initial condition $r(0)$ is considered now as a stochastic variable, called h^2, which accounts for the initial fluctuations responsible for dynamical relaxation of the system towards its steady (deterministic) state characterized by $r(t=\infty)=r_{\rm st}$. Therefore the NLRT (3.1) associated with the moment $\langle r\rangle$ will be given by the average of a quadrature

$$T=\frac{1}{r_{\rm st}}\left\langle \int_{r_0=h^2}^{r_{\rm st}}(r_{\rm st}-r)\frac{dr}{v(r)}\right\rangle . \tag{3.2}$$

By using Eq. (2.2), one can write this NLRT as

$$T=\frac{1}{2a}\left\langle \ln\left[\frac{r_{\rm st}}{h^2}\right]\right\rangle+\frac{1}{r_{\rm st}}\left\langle \int_{r_0=h^2}^{r_{\rm st}}g(r)dr\right\rangle . \tag{3.3}$$

This result is the most general expression for the NLRT of the quantity $\langle r\rangle$. It exhibits in the first term the universal logarithmic dependence characteristic of the relaxation of an unstable state. The second term accounts explicitly for the nonlinearities of any model. This can be better understood if we take into account the explicit expression of the NLRT corresponding to the linear model, given by Eq. (A5) of the Appendix. Then

$$T=T_L+C_1+\frac{1}{r_{\rm st}}\left\langle \int_{r_0=h^2}^{r_{\rm st}}g(r)dr\right\rangle . \tag{3.4}$$

T_L and C_1 are quantities obtained in Appendix A.

On the other side, if the relaxing quantity is the moment $\langle r^l\rangle$, where $l=1,2,\ldots,$ and r is the square of the vector $\mathbf{r}=(x_1,\ldots,x_n)$. Then the NLRT will now read

$$T=\frac{1}{r_{\rm st}^l}\left\langle \int_{r_0=h^2}^{r_{\rm st}}(r_{\rm st}^l-r^l)\frac{dr}{v(r)}\right\rangle . \tag{3.5}$$

Again, taking into account the expression of the NLRT of the linear model in Eq. (5), we have

$$T=T_L+C_1 +\frac{1}{r_{\rm st}}\left\langle \int_{r_0=h^2}^{r_{\rm st}}\left[[1+S(r)]g(r)+C_0\frac{S(r)}{r}\right]dr\right\rangle , \tag{3.6}$$

where $S(r)$ is a polynomial, $S(r)=\sum_{k=1}^{l-1}(r/r_0)^k$. The universal logarithmic term is clearly contained in the linear time scale T_L. The last term accounts for the nonlinearities of the model, and the order of the relaxing moment. We can also note that the time scale (3.4) is a particular case of (3.6), if we make $n=1$ and $l=1$. In this case $S(r)$ is equal to zero.

IV. EXAMPLES

As an application of our theoretical results, we will study two examples. The first one concerns the Landau model (one and n variables) and the second one is a laser model.

A. Landau model for one and n variables

The one-variable Landau model is defined by the deterministic equation $\dot{x}=ax-bx^3$. For the square modulus $r=x^2$ for one variable or $r=x_1^2+\cdots+x_n^2$ for n variables of this model, Eq. (2.1) is then $\dot{r}=2ar-2br^2$, such that $r_{st}=a/b$. In this particular case we have $f(r)=C_0$ and $g(r)=0$. Therefore the NLRT associated with the moment $\langle r\rangle$, according to Eq. (3.4), reads

$$T=T_L+C_1 \ . \tag{4.1}$$

We now use result (A7) of the Appendix, and in the limit of small noise, such that $\alpha^2 r_{st}\gg 1$, we get for this time scale

$$T\simeq\frac{1}{2a}\{\ln(\alpha^2 r_{st})-\Psi(\tfrac{1}{2})\}+\mathcal{O}(D) \ , \tag{4.2}$$

where $\Psi(\frac{1}{2})=-\gamma-2\ln 2$ is the digamma function [13] and γ is the Euler constant [13].

In the case of n variables, the NLRT associated with the lth moment $\langle r^l\rangle$, according to Eq. (3.6), is given by

$$\begin{aligned}T=T_L+C_1+\frac{1}{2a}[\Psi(l)+\gamma] \\ -\frac{1}{a}\left[\frac{1}{2}\left[\frac{h^2}{r_{st}}\right]+\frac{1}{4}\left[\frac{h^2}{r_{st}}\right]^2\right. \\ \left.+\cdots+\frac{1}{2(l-1)}\left[\frac{h^2}{r_{st}}\right]^{2(l-1)}\right] \ .\end{aligned} \tag{4.3}$$

With the help of (2.12) we can observe that $\langle(h/r_{st})^k\rangle$ is a quantity proportional to $1/\alpha^k$, for $k\geq 2$. Therefore it can be neglected in the limit of small-noise intensity. So the NLRT associated with the lth moment $\langle r^l\rangle$ of the Landau model in the case of n variables is

$$T\simeq\frac{1}{2a}\left\{\ln(\alpha^2 r_{st})+\left[\gamma+\Psi(l)-\Psi\left[\frac{n}{2}\right]\right]\right\}+\mathcal{O}(D) \ . \tag{4.4}$$

This final result can be reduced to the expression (4.2), for $l=1$ and $n=1$.

B. A laser model

Lasers have become the prototype systems to study the transient dynamics of unstable states. We are interested now in the study of two different situations of a laser model. In the first case the laser is under the action of an external field E_e. The second case considers the same model but in the absence of the external field. Therefore the results of this last model will be obtained from the first one, just making $E_e=0$. The first model has been used to study the detection of very weak optical external signals in a laser [10–12,14,15]. The Langevin-type equation associated with the complex, scaled, and dimensionless laser field $E=E_1+iE_2$ in the presence of a complex external field E_e reads [15]

$$\frac{dE}{dt}=-kE+\frac{FE}{1+\dfrac{AI}{F}}+k_eE_e+\xi(t) \ , \tag{4.5}$$

where a complex Gaussian noise $\xi(t)$ has zero mean and correlation function

$$\langle\xi^*(t)\xi(t')\rangle=2D\delta(t-t') \ . \tag{4.6}$$

Here $I=|E|^2$ is the laser intensity, $\xi(t)$ is a white noise describing the internal fluctuations of intensity D, and k k_e, A, and F are other parameters of the model [15].

Now the linear model ($A=0$) corresponding to Eq. (4.5) is written in terms of the components of the electric field E, namely

$$\frac{dE_j}{dt}=aE_j+\xi_j(t)+k_eE_{ej}, \quad j=1,2 \tag{4.7}$$

where $a=(F-k)$. The formal solution of Eq. (4.7) is

$$E_j(t)=h_j(t)e^{at} \ , \tag{4.8}$$

where

$$h_j(t)=\int_0^t e^{-at'}\eta_j(t')dt' \ , \tag{4.9}$$

but now we have that $\eta_j(t)=\xi(t)+k_eE_{ej}$. The quantity $h_j(t)$ plays the role of the initial conditions for long times, as in the one-variable case. Then $h_j(\infty)=h_j$ will also be a random variable with a nonzero mean value $\langle h_j\rangle=k_eE_j$ and a variance $\langle h_j^2\rangle-\langle h_j\rangle^2=D/a=\sigma$. The marginal probability corresponding to the modulus $h=(h_1^2+h_2^2)^{1/2}$ is now [12]

$$P(h)=2\alpha^2 hI_0(b\alpha^2 h)e^{-\alpha^2(h^2+b^2)} \ , \tag{4.10}$$

where $\alpha^2=1/2\sigma$, $b=k_e|E_e|/a$, and $I_0(z)$ is the modified Bessel function of zero order [13]. For $b=0$, we recover Eq. (2.12) with $n=2$.

We remark now that the deterministic equation for the variable $r=I=|E|^2$ is the same for both models because in the QD approach, the noise and the external field are included in the variable h. Equation (5) is now written in the form

$$\dot{r}=\frac{2ar(r_{st}-r)}{r_{st}\left[1+\dfrac{A}{F}r\right]} \ . \tag{4.11}$$

Therefore, for this model $C_0=r_{st}/2a$ and $g(r)=r_{st}A/2aF$. The expression for the time scale (3.4) associated with the moment $\langle r\rangle=\langle I\rangle$ reads

$$T=T_L+C_1+\frac{1}{2a}\left[\frac{F}{k}-1+\frac{A}{F}\langle h^2\rangle\right] \ , \tag{4.12}$$

where now $r_{st}=I_{st}=Fa/kA$ and $\langle h^2\rangle=1/\alpha^2+b^2$. Then the corresponding NLRT for the laser model, in the absence of external field $b=0$ and for small-noise intensity, $\alpha^2 r_{st} \gg 1$, reads

$$T \simeq \frac{1}{2a}\left\{\ln(\alpha^2 r_{st}) - \Psi(1) - 1 + \frac{F}{k}\right\} + \mathcal{O}(D) . \tag{4.13}$$

On the other hand, the NLRT associated with the mean value of the laser intensity, under the action of the external signal and for small-noise intensity, is [see Eq. (A9)]

$$T \simeq \frac{1}{2a}\left\{\ln(\alpha^2 r_{st}) - \Psi(1) - 1 + \frac{F}{k} + \frac{A}{F}b^2 - [E_1(b^2\alpha^2) + \gamma + \ln(b^2\alpha^2)]\right\} + \mathcal{O}(D) . \tag{4.14}$$

For a very weak external field, the term Ab^2/F can be neglected.

V. CONCLUSIONS

We want to remark here that the method we have proposed allows us to characterize the decay of unstable states by means of the direct application of the QD approach. This method is simple and shows the universal character of the NLRT for the decay of an unstable state which is contained in the logarithmic term and which comes from the linear term. The nonlinear contribution is explicitly obtained by an integral in Eqs. (3.4) and (3.6).

On the other hand, the detection of weak optical signals in a laser, which has been studied in Refs. [11,12, and 15], becomes transparent in terms of the method proposed in this paper. This approach can be also used to study the problem with colored or non-Gaussian noise [6].

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APPENDIX: THE NLRT OF THE LINEAR MODEL

As in the relaxation of an unstable initial state, the linear approximation plays a central role. We present here the explicit calculation for this case.

The linear evolution equation of the deterministic process for the variable x is $\dot{x} = ax$, and for the square modulus, $r = x^2$, we have

$$\dot{r} = 2ar . \tag{A1}$$

The evolution stops when $r(t) = r_{st}$, r_{st} being the deterministic steady state of the nonlinear model. This is the way to avoid the explosion of the linear model for $t \to \infty$.

The solution of this process is clearly

$$r(t) = r(0)e^{2at} , \tag{A2}$$

and up to a time t_i, such that $r(t_i) = r_{st}$ and $r(0) = r_0$ is the initial condition of the problem. This time t_i is given by

$$t_i = \frac{1}{2a}\ln\left[\frac{r_{st}}{r_0}\right] , \tag{A3}$$

and it is a random variable because the initial condition is also a random variable, $r_0 = h^2$. Therefore the whole process $r(t)$ can be written explicitly as

$$r(t) = h^2 e^{2at}\theta(t_i - t) + r_{st}\theta(t - t_i) , \tag{A4}$$

and $\theta(t)$ is the step function.

According to the definition of the NLRT, Eq. (3.1), we can obtain for the linear model that

$$T_L = \frac{1}{2a}\left\langle \ln\left[\frac{r_{st}}{h^2}\right]\right\rangle - C_1 , \tag{A5}$$

where $C_1 = 1/2a - 1/2ar_{st}\langle h^2 \rangle$. We can note that the logarithmic term has the same structure, whether the system has one or n variables. Now the statistical average of the quantity $\langle \ln h^2 \rangle$ is done with the corresponding $P(h)$. We present now the explicit results for the linear approximation of the examples.

(i) *The Landau model for one and n variables.* The linear equation for both cases is given by $\dot{r} = 2ar$. Then the T_L is the same as (A5), but for one variable it can be calculated with the help of Eq. (2.8),

$$T_L = \frac{1}{2a}\{\ln(\alpha^2 r_{st}) - \Psi(\tfrac{1}{2})\} - C_1 , \tag{A6}$$

and for the n variables we use Eq. (2.12) to obtain

$$T_L = \frac{1}{2a}\left\{\ln(\alpha^2 r_{st}) - \Psi\left[\frac{n}{2}\right]\right\} - C_1 . \tag{A7}$$

(ii) *The laser model.* For this model the linear deterministic equation is again $\dot{r} = 2ar$, but $a = (F - k)$, where $r = |E|^2$. Therefore, the T_L for the model with the parameter $b = 0$, according to Eq. (4.10), is given by

$$T_L = \frac{1}{2a}\{\ln(\alpha^2 r_{st}) - \Psi(1)\} - C_1 . \tag{A8}$$

Finally, for the laser model in the presence of the external field E_e, $b \neq 0$, we can obtain T_L again using Eq. (4.10),

$$T_L = \frac{1}{2a}\{\ln(\alpha^2 r_{st}) - \Psi(1) - [E_1(b^2\alpha^2) + \gamma + \ln(b^2\alpha^2)]\} - C_1 , \tag{A9}$$

where $E_1(x)$ is the integral exponential function [13].

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